The Elastic Lateral Stability of Trusses

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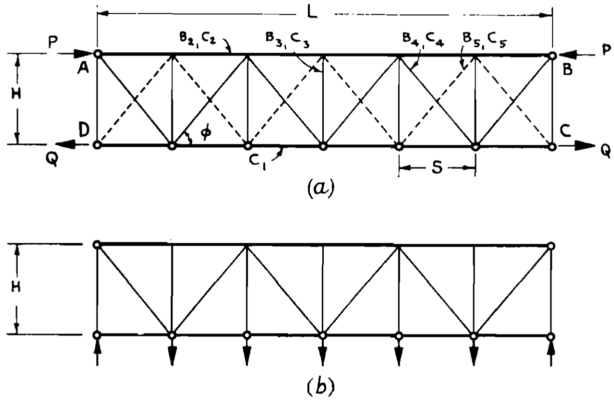
Synopsis

The article gives an approximate analysis, using the energy method, for the elastic stability out of their plane of trusses with outstanding compression chords, the tension chord being held in position. Previous discussions of this problem have usually considered it in terms of the stability of the compression chord as a member subjected to various elastic restraints. In the present treatment, the truss is considered as a whole, account being taken of the flexural and torsional rigidities of all the members, and of partial restraint of the tension chord against twisting. The analysis is interpreted in relation to a number of common arrangements of vertical and diagonal web members. The analysis is derived for a truss in which the compression chord is of uniform section, carrying a uniform thrust, but the use of some of the results as approximate criteria of stability for non-uniform conditions is also discussed.

Introduction

This article deals with the lateral stability of an elastic plane truss with parallel chords, such as that shown in Fig. 1(a). The analysis applies to any arrangement of vertical and diagonal web members, the only restriction being that no account is taken of intersections within the web. The truss is subjected to any combination of uniform bending moment and overall axial load. The compression chord (AB in Fig. 1(a)) sustains an axial load P and is laterally supported only by the web members of the truss, except at A and B, where it is assumed to be restrained against deflection out of the plane ABCD. The chord CD may carry any axial load Q (tensile or compressive), and is laterally supported at the panel points. Allowance is made in the analysis for elastic restraints against twisting of the chord CD about its longitudinal axis, these restraints being applied at panel points. Although this simplified form of loading is assumed, some of the results may be applied approximately to a truss more realistically loaded as shown in Fig. 1(b) provided the vertical loads are applied to the lower chord only, and the web members do not sustain compressive axial loads which are a high proportion of their axial loading capacities as pin-ended struts. The analysis is of particular interest in relation to trusses composed of tubular members, but is not restricted to such cases. The joints are all assumed to be rigid, and the members prismatic between panel points. For the sake of analysis it is also assumed that the chords are each of uniform section, while each set of web members (for example, the verticals) is of uniform section throughout the truss.

When the axial load P in the compression chord reaches a critical value, the truss buckles laterally as shown in Fig. 2(a). The buckling mode may be in a single half-wave, or in a series of almost equal half-waves. The compression chord is acted upon by a series of torques, moments and shear forces from the web members.



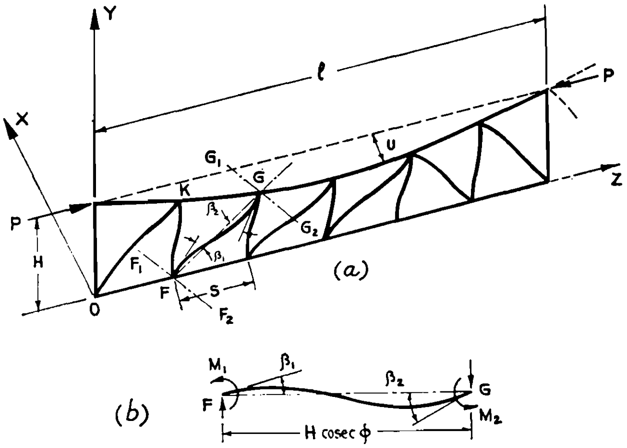
o LATERAL RESTRAINTS.

TRUSS UNDER (a) UNIFORM THRUST AND TENSION.

(b) VARYING BENDING MOMENT.

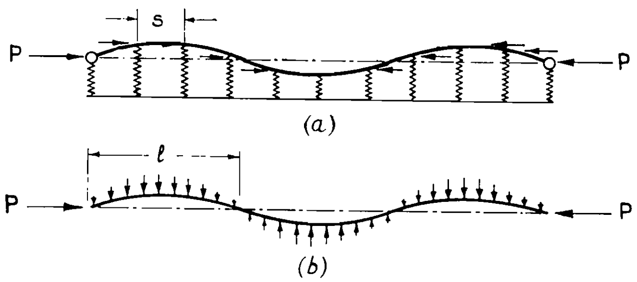
FIG. 1.

In most treatments of this problem, the stability of the compression chord is investigated by assuming that the restraints from the web members may be represented by the action of springs lying in a horizontal plane, as shown in Fig. 3(a). Bleich[1] suggests methods for solving this idealisation of the problem, allowing for changing axial load and cross-section of the compression chord and variations in the spacing and stiffness of the springs. For the simpler case of a uniform axial load in a chord of uniform section, with equal spacing and stiffness of the springs, an early approximate solution by Engesser[2] gives good results. If the springs in Fig. 3(a) require a force k to produce unit deflection, and the springs are



TRUSS IN THE BUCKLED STATE.

FIG. 2.



STABILITY OF ELASTICALLY SUPPORTED BAR.

FIG. 3.

TABLE 1		
MEMBER	FLEXURAL RIGIDITY	TORSIONAL RIGIDITY.
TENSION BOOM	–	C_1
COMPRESSION BOOM	B_2	C_2
VERTICALS	B_3	C_3
DIAGONALS	B_4, B_5	C_4, C_5

$B = (B_4 + B_5)\ \sin^3\phi$
$C = (C_4 + C_5)\ \sin^3\phi$
$A = \frac{H^2}{S} \times$ {FLEXURAL STIFFNESS OF TRANSVERSE MEMBERS AT EACH LOWER PANEL POINT.}

spaced at intervals s, Engesser replaces the springs by a continuous elastic medium which resists unit displacement by a force of $\frac{k}{s}$ per unit length of the chord, as shown in Fig. 3(b). It is then found that the chord buckles in half-waves of length l where

$$l = \pi\left(\frac{EIs}{k}\right)^{\frac{1}{4}} \qquad \cdots \qquad (1)$$

in which EI is the flexural rigidity of the chord for bending out of the plane of the truss. The critical value of the axial load P is

$$P = 2\sqrt{\frac{EIk}{s}} \qquad \cdots \qquad (2)$$

Engesser showed experimentally that equation (2) is remarkably accurate provided $\frac{l}{s} > 1{\cdot}8$ and this has been confirmed theoretically by Bleich[1]. Equation (2) has been used extensively for bridge trusses in which the web members in Fig. (1) are almost completely direction fixed at their feet. The Engesser solution assumes that the chord is infinitely long and that the ends are fixed against lateral deflection with sufficient rigidity for the buckling load not to be reduced by buckling at the ends. The effect of completely free ends has been discussed by Zimmermann[3], while Chwalla[4] considers ends of finite rigidity. The case of chords of finite length in a continuous elastic medium, but with ends fixed against lateral deflection, has been discussed by Timoshenko[5], who deals not only with uniform axial loads in the chord, but also with axial loads varying parabolically.

Reference to Fig. 2 shows that the compression chord in a truss is not only restrained against lateral deflection (in direction OX), but that the chord members also provide restraint against the twisting of the chord about its longitudinal axis (parallel to OZ). The rotation of the chord at panel points about axes parallel to OY is also partially restrained, this restraint being due to twisting in the vertical chord members and to bending and twisting in the diagonal members. The buckling of a uniformly compressed, uniform chord member with equally spaced lateral and rotational restraints has been discussed by Budiansky *et al*[6]. For the compression chord in Fig. 2, this treatment would allow for restraints at panel points against deflections parallel to OX and for restraint against rotation about axes parallel to OY, but would ignore the effects associated with the twisting of the chord about its own longitudinal axis. The effect of bending moment in the vertical members, and their interaction with the twisting of the chord has been discussed by Hrennikoff[7] who, however, ignores some of the other restraints.

An exhaustive study of the problem of buckling of chords in pony trusses has recently been conducted by Holt[8]. It will be appreciated that a complete solution is extremely involved, and Holt's treatment is too complicated for practical use. In a review of theoretical and experimental work on the buckling of top chords in pony trusses, Handa[9] comes to the conclusion that the simple formula of Engesser gives results as good as any for the collapse loads of actual trusses, and that the complications of most treatments are not worthwhile. This is perhaps not surprising in view of Bleich's findings in relation to Engesser's formula. The trusses considered by Handa are composed of open section members, so that effects associated with the resistance of the chord and web members to twisting are of no importance provided local torsional buckling does not occur. When closed sections are used, and buckling rigidities become of the same order as flexural rigidities, the Engesser formula can no longer be expected to give reasonable results. The essential features of the Engesser solution may, however, be retained by considering all restraints to the chord from the web members to be distributed continuously as in a special sort of elastic medium. It is thus possible to make a sufficiently accurate allowance for all the restraints which have been considered in a more elaborate manner by the many authors who have written on the subject. This generalised solution to the buckling of a compression chord in a truss is the subject of the present paper.

It is a common feature of the treatments previously given that they consider the feet of the web members to be completely or partially restrained, without any allowance for the behaviour of the tension chord. Because of the high rotational restraint at the feet of the web members in bridge trusses, the neglect of the tension chord is justified, but trusses forming part of a building structure may be in a different category. Subsidiary members such as sheeting rails, connected to the tension chord, will usually suffice to restrain it in position laterally, but may not have large enough flexural stiffness to offer significant restraint against twisting. If the tension chord is of tubular section, it will then contribute appreciably to the stability of the truss. A solution which allows for the resistance to twisting of the tension chord must necessarily treat the truss as a whole, and this is the basis of what follows. The analysis is based on the energy method, allowance being made for the resistance to bending and twisting of all the members, and also for partial twisting restraint applied to the tension chord. The analytical results are summarised in Table 2 on pages 150 and 151. The solutions are interpreted specifically in relation to six arrangements of vertical and diagonal members, but are not restricted to these arrangements.

Notation

The notation is summarised in Figs. 1(a) and 2 and in Table 1. The flexural rigidities B_2, B_3, B_4 and B_5 are the values of EI for bending out of the plane of the truss (E modulus of elasticity, I moment of inertia). The torsional rigidities C_1 to C_5 are the values of GJ (G elastic shear modulus and J the St. Venant torsion constant). The flexural stiffness of the members (not shown) which restrain the tension chord CD against twisting is defined by the quantity A, which has the dimensions of the flexural and torsional rigidities ((force) $\times$ (distance)^2). If the external members attached to the tension chord produce a torque of T at each panel point for unit angle of twist of the chord, then $A = \frac{H^2}{s} T$ where H is the depth between centres of chords and s is the distance between panel points (Fig. 1(a)). The symbol $\varnothing$ denotes the angle between the diagonal members and the tension and compression chords. The length of the truss, between points at which the compression chord is restrained against lateral displacement, is denoted by L, while l is the half-wavelength for the compression chord in the buckled state. The thrust in the compression chord is denoted by P.

Various symbols, (μ, η, B, C, D, F) are used to denote functions of the above quantities, and are defined as occasion arises in Table 2. In the analysis we take axis OZ along the centre of the tension chord (Fig. 2(a)), OY in the plane of the truss perpendicular to OZ, and OX perpendicular to OY and OZ. The angle of twist of the tension chord is denoted by θ, and of the compression chord by θ_2, while the deflection of a point on the compression chord out of the plane OYZ is denoted by u. Taking any particular web member FG in the deformed state (Fig. 2(b)), the angles between the tangents at F and G and the straight line FG are denoted by β_1 and β_2 respectively.

General Analysis

It will be assumed that, in the buckled state, the lateral deflection of the compression chord (in direction XO) is given by

$$u = H\theta \sin\frac{\pi z}{l} \qquad \ldots \quad (3)$$

where z is measured from one end. It is assumed that the tension chord remains straight. These deflected forms neglect the local distortions produced in the chords by the bending and twisting resistance of the web members, and are justified provided the flexural rigidities of the chords are large compared with the flexural and torsional rigidities of the web members. The angles of twist of the tension and compression chords (clockwise about OZ) are represented by

$$\theta_1 = a_1\theta \sin\frac{\pi z}{l}, \qquad \ldots \quad (4)$$

$$\theta_2 = a_2\theta \sin\frac{\pi z}{l}. \qquad \ldots \quad (5)$$

The external work, per length l of the truss, due to the thrust P in the compression chord is U_P where

$$U_P = \frac{P}{2}\int_0^l \left(\frac{du}{dz}\right)^2 dz = \frac{\pi^2}{4}\frac{PH^2}{l}\theta^2. \qquad \ldots \quad (6)$$

Since the tension chord is assumed to remain straight no work is done by the force Q. The external work U_P has to be equated to the total strain energy due to buckling in the members of the truss.

The tension chord has strain energy due to twisting U_{C1} where

$$U_{C1} = \frac{C_1}{2}\int_0^l \left(\frac{d\theta_1}{dz}\right)^2 dz = \frac{\pi^2 a_1^2}{4}\frac{C_1}{l}\theta^2. \qquad \ldots \quad (7)$$

The compression chord has strain energies U_{B2} due to bending and U_{C2} due to twisting where

$$U_{B2} = \frac{B_2}{2}\int_0^l \left(\frac{d^2u}{dz^2}\right)^2 dz = \frac{\pi^4}{4}\frac{B_2H^2}{l^3}\theta^2, \qquad \ldots \quad (8)$$

$$U_{C2} = \frac{C_2}{2}\int_0^l \left(\frac{d\theta_2}{dz}\right)^2 dz = \frac{\pi^2 a_2^2}{4}\frac{C_2}{l}\theta^2. \qquad \ldots \quad (9)$$

We consider now the twisting and flexure of any diagonal member FG (Fig. 2). The line F_1F_2 is taken through F perpendicular to FG and in the plane OYZ. The line G_1G_2 passes through G and is parallel to F_1F_2. Since there is continuity between the tension chord and diagonal FG at F, the tangent to FG rotates during buckling about F_1F_2 through the angle $\left(a_1\theta \sin\frac{\pi z}{l}.\sin\varnothing\right)$ where z is the distance of F from the origin. Similarly, the tangent to FG at G rotates about G_1G_2 through the angle

$$\left(a_2\theta \sin\frac{\pi(z+s)}{l}.\sin\varnothing + \frac{\pi H}{l}\theta\cos\frac{\pi(z+s)}{l}.\cos\varnothing\right)$$

The chord FG rotates about F_1F_2 through the angle $\left(\theta \sin\frac{\pi(z+s)}{l}.\sin\varnothing\right)$. The angles β_1 and β_2 in Fig. 2(b) are thus

$$\beta_1 = \theta\left[\sin\frac{\pi(z+s)}{l} - a_1\sin\frac{\pi z}{l}\right]\sin\varnothing,$$

$$\beta_2 = \theta\left[\left(\sin\frac{\pi(z+s)}{l} - a_2\sin\frac{\pi(z+s)}{l}\right)\sin\varnothing - \frac{\pi H}{l}\cos\frac{\pi(z+s)}{l}.\cos\varnothing\right].$$

We now make the assumption that H and s are small compared with l, and ignore s in comparison with z. Hence

$$\beta_1 = (1-a_1)\theta\sin\frac{\pi z}{l}.\sin\varnothing, \qquad \ldots \quad (10)$$

$$\beta_2 = (1-a_2)\theta\sin\frac{\pi z}{l}.\sin\varnothing - \frac{\pi H}{l}\theta\cos\frac{\pi z}{l}.\cos\varnothing. \qquad \ldots \quad (11)$$

If we denote the bending moments at F and G in the member FG by M_1 and M_2 respectively as shown in Fig. 2(b), then the slope-deflection equations give

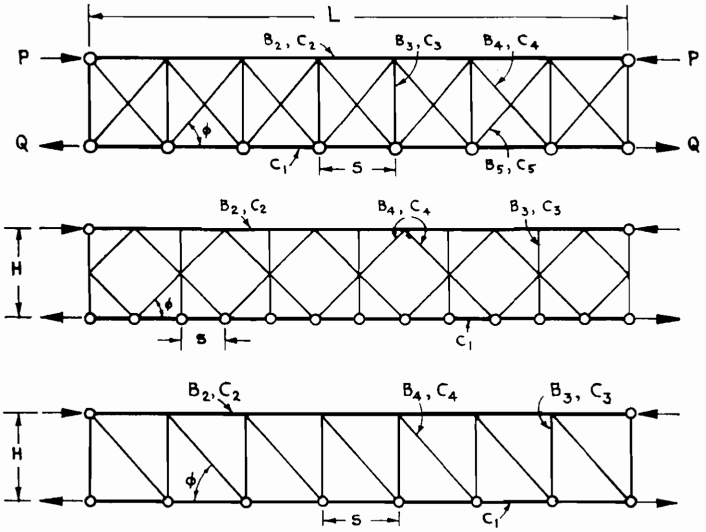
$$M_1 = \frac{2B_4}{H}(2\beta_1 + \beta_2)\sin\varnothing, \qquad \ldots \quad (12)$$

$$M_2 = \frac{2B_4}{H}(\beta_1 + 2\beta_2)\sin\varnothing \qquad \ldots \quad (13)$$

where B_4 is the flexural rigidity of FG. The flexural energy in FG is this ΔU_{B4} where

$$\Delta U_{B4} = \tfrac{1}{2}(M_1\beta_1 + M_2\beta_2) = \frac{2B_4}{H}(\beta_1^2 + \beta_1\beta_2 + \beta_2^2)\sin\varnothing. \qquad \ldots \quad (14)$$

TABLE 2.



$B = (B_4 + B_5)\sin^3\phi$
$C = (C_4 + C_5)\sin^3\phi$

$B = 2B_4 \sin^3\phi$
$C = 2C_4 \sin^3\phi$

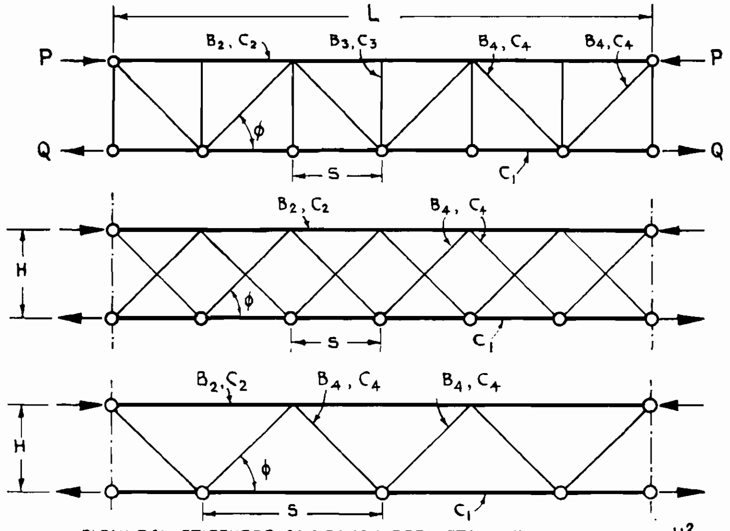
$B = B_4 \sin^3\phi$
$C = C_4 \sin^3\phi$

O DENOTES COMPLETE RESTRAINT AGAINST LATERAL DEFLECTION.

GENERAL SOLUTION.	$D = C\cot\phi + (B+B_3)\tan\phi, \quad F = C\cot\phi + 4(B+B_3)\tan\phi.$ LET $n = \left(\frac{L}{m\pi H}\right)$ WHERE $m = 1, 2, \text{------------},$ $a_1 = \frac{6(B+B_3)\left(2D + \frac{C_2}{n^2}\right)}{\left(A+F+\frac{C_1}{n^2}\right)\left(F+\frac{C_2}{n^2}\right) + 9C(B+B_3) - DF},$ $a_2 = \frac{6(B+B_3)\left(A + 2D + \frac{C_1}{n^2}\right)}{\left(A+F+\frac{C_1}{n^2}\right)\left(F+\frac{C_2}{n^2}\right) + 9C(B+B_3) - DF},$ $P = \frac{1}{H^2}\left[\frac{B_2}{n^2} + a_1^2 C_1 + a_2^2 C_2 + 4B\cot\phi + (C+C_3)\tan\phi + n^2\left\{a_1^2 A + 4\left[(1-a_1)^2 + (1-a_1)(1-a_2) + (1-a_2)^2\right](B+B_3)\tan\phi + (a_1-a_2)^2 C\cot\phi\right\}\right].$ TAKE SMALLEST P $(m = 1, 2, \text{--------})$.
$A = 0$ $C_1 = C_2$	$\mu^2 = 12\tan\phi.$ $\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1} < 1$ AND $\left(\frac{L}{H}\right)^2 > \frac{\pi^2}{\mu}\sqrt{\frac{B_2}{B+B_3}}\;\frac{1}{1-\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1}},$ $P = \frac{1}{H^2}\left[2\mu\sqrt{B_2(B+B_3)}\left\{1-\frac{\mu}{4}\frac{\sqrt{B_2(B+B_3)}}{C_1}\right\} + 4B\cot\phi + (C+C_3)\tan\phi\right].$ $\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1} < 1$ AND $\left(\frac{L}{H}\right)^2 < \frac{\pi}{\mu}\sqrt{\frac{B_2}{B+B_3}}\;\frac{1}{1-\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1}}$ * OR $\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1} > 1$ (ALL VALUES OF L), $P = \pi^2\frac{B_2}{L^2} + \frac{1}{H^2}\left[\frac{2C_1}{1+\frac{2\pi^2}{\mu^2}\left(\frac{H}{L}\right)^2\frac{C_1}{B+B_3}} + 4B\cot\phi + (C+C_3)\tan\phi\right].$ SAFE RESULT, ALL LENGTHS, $\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1} < 1,$ $P = \frac{1}{H^2}\left[2\mu\sqrt{B_2(B+B_3)}\left\{1-\frac{\mu}{4}\frac{\sqrt{B_2(B+B_3)}}{C_1}\right\} + 4B\cot\phi + (C+C_3)\tan\phi\right].$ SAFE RESULT, ALL LENGTHS, $\frac{\mu}{2}\frac{\sqrt{B_2(B+B_3)}}{C_1} > 1,$ $P = \frac{1}{H^2}\left[2C_1 + 4B\cot\phi + (C+C_3)\tan\phi\right].$

* For $\frac{\pi}{\mu}$ read $\frac{\pi^2}{\mu}$

TABLE 2 CONTINUED.



$B = B_4 \sin^3\phi$, $C = C_4 \sin^3\phi$

$B = 2B_4 \sin^3\phi,\ B_3 = 0$; $C = 2C_4 \sin^3\phi,\ C_3 = 0$

$B = B_4 \sin^3\phi,\ B_3 = 0$; $C = C_4 \sin^3\phi,\ C_3 = 0$

T = FLEXURAL STIFFNESS OF TRANSVERSE ATTACHMENTS AT EACH LOWER PANEL POINT. $A = \frac{H^2}{S} T$.

$A = 0$, $C_1 = 0$

$$\eta^2 = 12 \tan\phi \left\{ \frac{C\cot\phi + (B+B_3)\tan\phi}{C\cot\phi + 4(B+B_3)\tan\phi} \right\}.$$

$\eta \frac{\sqrt{B_2(B+B_3)}}{C_2} < 1$ AND $\left(\frac{L}{H}\right)^2 > \frac{\pi^2}{\eta}\sqrt{\frac{B_2}{B+B_3}} \frac{1}{1-\eta\sqrt{B_2(B+B_3)}/C_2}$,

$$P = \frac{1}{H^2}\left[2\eta\sqrt{B_2(B+B_3)}\left\{1 - \frac{\eta}{2}\frac{\sqrt{B_2(B+B_3)}}{C_2}\right\} + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

$\eta \frac{\sqrt{B_2(B+B_3)}}{C_2} < 1$ AND $\left(\frac{L}{H}\right)^2 < \frac{\pi^2}{\eta}\sqrt{\frac{B_2}{B+B_3}} \frac{1}{1-\eta\sqrt{B_2(B+B_3)}/C_2}$,

OR $\eta \frac{\sqrt{B_2(B+B_3)}}{C_2} > 1$ (ALL VALUES OF L),

† $$P = \pi^2 \frac{B_2}{L^2} + \frac{1}{H^2}\left[\frac{C_2}{1 + \frac{\pi}{\eta^2}\left(\frac{H}{L}\right)^2 \frac{C_2}{B+B_3}} + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

SAFE RESULT, ALL LENGTHS, $\eta\frac{\sqrt{B_2(B+B_3)}}{C_2} < 1$,

$$P = \frac{1}{H^2}\left[2\eta\sqrt{B_2(B+B_3)}\left\{1 - \frac{\eta}{2}\frac{\sqrt{B_2(B+B_3)}}{C_2}\right\} + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

SAFE RESULT, ALL LENGTHS, $\eta\frac{\sqrt{B_2(B+B_3)}}{C_2} > 1$,

$$P = \frac{1}{H^2}\left[C_2 + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

$A = 0$, $C_2 = 0$

AS $\left\{\begin{matrix} A = 0 \\ C_1 = 0 \end{matrix}\right\}$ WITH C_2 REPLACED BY C_1.

$C_1 = 0$, $C_2 = 0$

$$\eta^2 = 12 \tan\phi \left\{ \frac{C\cot\phi + (B+B_3)\tan\phi}{C\cot\phi + 4(B+B_3)\tan\phi} \right\}.$$

$\left(\frac{L}{H}\right)^2 > \frac{\pi^2}{\eta}\sqrt{\frac{B_2}{B+B_3}}\sqrt{1 + \eta^2\frac{B+B_3}{A}}$,

$$P = \frac{1}{H^2}\left[\frac{2\eta\sqrt{B_2(B+B_3)}}{\sqrt{1+\eta^2\frac{B+B_3}{A}}} + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

$\left(\frac{L}{H}\right)^2 < \frac{\pi^2}{\eta}\sqrt{\frac{B_2}{B+B_3}}\sqrt{1 + \eta^2\frac{B+B_3}{A}}$,

$$P = \pi^2\frac{B_2}{L^2} + \frac{1}{H^2}\left[\frac{\frac{\eta^2}{\pi^2}\left(\frac{L}{H}\right)^2(B+B_3)}{1 + \eta^2\frac{B+B_3}{A}} + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

SAFE RESULT, ALL LENGTHS,

$$P = \frac{1}{H^2}\left[\frac{2\eta\sqrt{B_2(B+B_3)}}{\sqrt{1+\eta^2\frac{B+B_3}{A}}} + 4B\cot\phi + (C+C_3)\tan\phi\right].$$

† For $\frac{\pi}{\eta^2}$ read $\frac{\pi^2}{\eta^2}$

If there is one diagonal number for each length s of the truss (as in Fig. 2), the mean bending energy in the diagonals per unit length of the truss is

$$\frac{\triangle U_{B4}}{s} = \frac{\triangle U_{B4}}{H} \tan \varnothing .$$

The total bending energy in the diagonals throughout the length l is

$$U_{B4} = \int_0^l \left(\frac{\triangle U_{B4} \tan \varnothing}{H} \right) dz. \quad . \quad . \quad . \quad (15)$$

Substituting for β_1 and β_2 from equations (10) and (11) in equation (14), and thence for $\triangle U_{B4}$ in equation (15), the value of U_{B4} becomes, on integration,

$$U_{B4} = \left[\left\{ (1-a_1)^2 + (1-a_1)(1-a_2) + (1-a_2)^2 \right\} \tan \varnothing + \left(\frac{\pi H}{l} \right)^2 \cot \varnothing \right] \frac{(B_4 \sin^3 \varnothing) l}{H^2} \cdot \theta^2. \quad . \quad (16)$$

The end F of the diagonal FG twists about FG through the angle $\left(a_1 \, \theta \sin \frac{\pi z}{l} . \cos \varnothing \right)$, while end G twists through the angle

$$\left(a_2 \, \theta \sin \frac{\pi(z+s)}{l} . \cos \varnothing - \frac{\pi H}{l} \theta \cos \frac{\pi(z+s)}{l} . \sin \varnothing \right)$$

Again, neglecting s in comparison with z, the twisting energy $\triangle U_{C4}$ in the diagonal is

$$\triangle U_{C4} = \frac{C_4}{2H} \left\{ (a_1 - a_2) \, \theta \sin \frac{\pi z}{l} \cdot \cos \varnothing + \frac{\pi H}{l} \theta \cos \frac{\pi z}{l} \cdot \sin \varnothing \right\}^2 \sin \varnothing . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The total twisting energy in the diagonals is U_{C4} where

$$U_{C4} = \int_0^l \left(\frac{\triangle U_{C4} \tan \varnothing}{H} \right) dz, \quad . \quad . \quad . \quad (18)$$

whence

$$U_{C4} = \left[(a_1 - a_2)^2 \cot \varnothing + \left(\frac{\pi H}{l} \right)^2 \tan \varnothing \right] \times \frac{(C_4 \sin^3 \varnothing) l}{4H^2} \theta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The bending and twisting energies in the vertical members of the truss may be derived from the values for the diagonal members as follows. It is assumed that the vertical members are spaced at intervals of $s = H \cot \varnothing$, as in Fig. 2(a). The angles β_1 and β_2 for any vertical FK are obtained by putting $\varnothing = \frac{\pi}{2}$ in equations (10) and (11), and the bending energy $\triangle U_{B3}$ is then obtained from equation (14) by putting $\varnothing = \frac{\pi}{2}$ and replacing B_4 by B_3. The total bending energy U_{B3} in all the verticals is obtained from equation (15), but $\tan \varnothing$ must here be retained since it corresponds to $\frac{H}{s}$ and defines the spacing of the members.

Hence

$$U_{B3} = \left\{ (1-a_1)^2 + (1-a_1)(1-a_2) + (1-a_2)^2 \right\} \times \tan \varnothing . \frac{B_3 l}{H^2} \theta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Similarly we obtain the twisting energy U_{C3} in the verticals,

$$U_{C3} = \left(\frac{\pi H}{l} \right)^2 \tan \varnothing . \frac{C_3 l}{4H^2} \theta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Finally we require the energy stored in the subsidiary members which restrain the tension chord against twisting. The restraining torque per unit length is $\frac{T}{s} a_1 \theta \sin \frac{\pi z}{l}$, and since $\frac{T}{s} = \frac{A}{H^2}$, the total restraining energy U_A over the buckling length l is

$$U_A = \frac{A}{2H^2} \int_0^l \left(a_1 \theta \sin \frac{\pi z}{l} \right)^2 dz = a_1^2 \frac{Al}{4H^2} \theta^2. \quad . \quad (22)$$

The energy equation for the length l of the truss is

$$U_P = U_A + U_{B2} + U_{B3} + U_{B4} + U_{C1} + U_{C2} + U_{C3} + U_{C4}. \quad . \quad . \quad (23)$$

Upon substituting for the expressions for the separate terms, we have

$$PH^2 = a_1^2 C_1 + a_2^2 C_2 + 4B \cot \varnothing + (C + C_3) \tan \varnothing + \frac{B_2}{n^2} + n^2 a_1^2 A + n^2 (a_1 - a_2)^2 C \cot \varnothing + 4n^2 [(1-a_1)^2 + (1-a_1)(1-a_2 + (1-a_2)^2] (B + B_3) \tan \varnothing$$

where . . . . (24)

$$n = \frac{l}{\pi H},$$
$$B = B_4 \sin^3 \varnothing,$$
$$C = C_4 \sin^3 \varnothing.$$

The buckling length l may assume any value $\frac{L}{m}$ where m is a digit. For each l, the corresponding buckling load P is obtained by putting $\frac{\partial P}{\partial a_1} = 0$ and $\frac{\partial P}{\partial a_2} = 0$, so that P is a minimum with respect to the arbitrarily chosen coefficients a_1 and a_2.

Hence

$$2(3 - 2a_1 - a_2)(B + B_3) \tan \varnothing - (a_1 - a_2) C \cot \varnothing - a_1 A = \frac{a_1 C_1}{n^2}, \quad . \quad . \quad (25)$$

$$2(3 - a_1 - 2a_2)(B + B_3) \tan \varnothing + (a_1 - a_2) C \cot \varnothing = \frac{a_2 C_2}{n^2}. \quad . \quad . \quad . \quad (26)$$

Equations (25) and (26) may be solved for a_1 and a_2 and the resulting values of a_1 and a_2 substituted in equation (24). If $P = P_1$ corresponds to $m = 1$, $P = P_2$ to $m = 2$ etc., it will usually be most convenient first to calculate P_1 and P_2; if $P_1 < P_2$, then P_1 is the required solution. If $P_1 > P_2$, then P_3 is calculated; if then $P_2 < P_3$, P_2 is the solution, and so on. The complete equations for this general solution are given at the beginning of Table 2 on page 150. The diagrams at the head of the table show how the results may be applied to trusses with some common arrangements of web members by suitably adjusting the significance of the terms B and C.

The above procedure is lengthy, and in some cases a more convenient solution, on the safe side, may be obtained if it is assumed that l may possess any value less than L. We then have $\frac{\partial P}{\partial l} = 0$ or, since $n = \frac{l}{\pi H}$, $\frac{\partial P}{\partial n} = 0$.

Hence

$$a_1^2 A + (a_1 - a_2)^2 C \cot \varnothing + 4[(1 - a_1)^2 + (1 - a_1)(1 - a_2) + (1 - a_2)^2](B + B_3) \tan \varnothing = \frac{B_2}{n^4}. \quad (27)$$

The value of P now appears as the solution of equations (24) to (27), provided $l < L$ or $n < \frac{L}{\pi H}$. The general solution for P is not explicit, but an expression for P may be obtained for the following special cases.

(1) $A = 0, \quad C_1 = C_2,$
(2) $A = 0, \quad C_1 = 0,$
(3) $A = 0, \quad C_2 = 0,$
(4) $C_1 = 0, \quad C_2 = 0.$

Results for these four sets of conditions are summarised in Table 2. Their derivation will be illustrated by considering the second case, $A = 0$, $C_1 = 0$.

The Case A = 0 C₁ = 0

If we put $\gamma = \left(\frac{B + B_3}{C}\right) \tan^2 \varnothing$,

equation (25) becomes

$$(1 - a_1) = \frac{1 - 2\gamma}{1 + 4\gamma}(1 - a_2) \quad (28)$$

and if $\eta^2 = 12 \tan \varnothing \frac{1 + \gamma}{1 + 4\gamma}$, equations (26) and (27) become respectively

$$n^2 \eta^2 \frac{(1 - a_2)}{a_2} = \frac{C_2}{B + B_3}, \quad (29)$$

$$n^4 \eta^2 (1 - a_2)^2 = \frac{B_2}{B + B_3} \quad (30)$$

Hence

$$a_2 = \frac{\eta \sqrt{B_2(B + B_3)}}{C_2}, \quad (31)$$

$$n^2 = \frac{1}{\eta}\sqrt{\frac{B_2}{B + B_3}} \; \frac{1}{1 - \eta \frac{\sqrt{B_2(B + B_3)}}{C_2}} \quad (32)$$

By substituting for a_1, a_2 and n in equation (24), we obtain

$$P = \frac{1}{H^2}\left[2\eta\sqrt{B_2(B + B_3)}\left\{1 - \frac{\eta}{2}\frac{\sqrt{B_2(B + B_3)}}{C_2}\right\} + 4 B \cot \varnothing + (C + C_3) \tan \varnothing\right] \quad (33)$$

Equation (33) applies provided the length $l = n\pi H$ obtained from equation (32) is less than the length of the truss L, that is provided

$$\left(\frac{L}{H}\right)^2 > \frac{\pi^2}{\eta}\sqrt{\frac{B_2}{B + B_3}} \; \frac{1}{1 - \eta \frac{\sqrt{B_2(B + B_3)}}{C_2}}. \quad (34)$$

When L is smaller than the limit given by the inequality (34), the buckling length l must be taken equal to L, and the value of P is obtained by substituting $n = \frac{L}{\pi H}$ in equations (24), (25) and (26). Hence it is found that

$$P = \pi^2 \frac{B_2}{L^2} + \frac{1}{H^2}\left[\frac{C_2}{1 + \frac{\pi^2}{\eta^2}\left(\frac{H}{L}\right)^2 \frac{C_2}{B + B_3}} + 4 B \cot \varnothing + (C + C_3) \tan \varnothing\right]. \quad (35)$$

A safe solution for P will result if the member is assumed to be infinitely long, whatever its actual length may be. For such a safe solution, equation (33) is appropriate provided the value of n given by equation (32) is real, that is, provided

$$\eta \frac{\sqrt{B_2(B + B_3)}}{C_2} < 1.$$

When this condition is not satisfied, it denotes that the buckling length is infinitely long, and a safe value for P is obtained from equation (35) by putting $L = \infty$, that is

$$P = \frac{1}{H^2}\left[C_2 + 4B \cot \varnothing + (C + C_3) \tan \varnothing\right]. \quad (36)$$

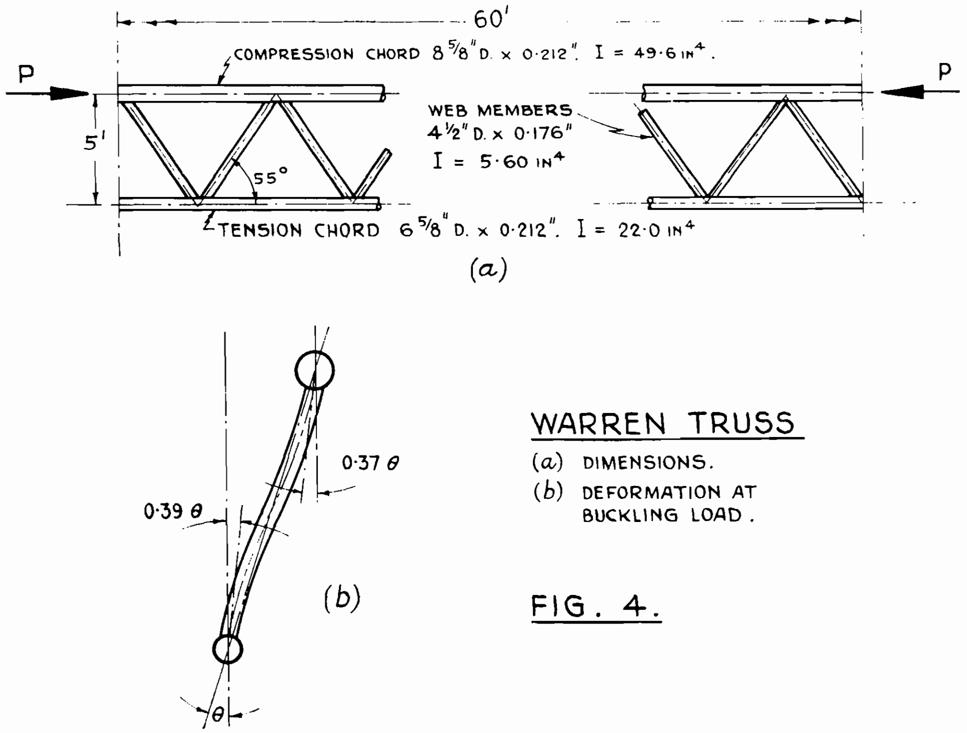
The interesting feature of equations (33) and (36) is that they provide a safe estimate of the buckling load of the truss without reference to the length between laterally restrained points. These safe estimates are given for all the special cases in Table 2.

Numerical Example

The Warren truss in Fig. 4(a), 60 feet long and 5 feet deep between chord centres, is composed of circular steel tubes of the sections indicated. The modulus of elasticity, E, is 13,000 tons/in.² and the elastic shear modulus, G, is 5,000 tons/in.² Each panel point on the lower chord is partially restrained against twisting about the longitudinal axis to give $A = 150 \times 10^3$ tons in²., corresponding to a stiffness at each panel point of 3,500 tons ins.

The value of P which would cause instability, all members being assumed to remain elastic, may be estimated in various ways, and the solutions are summarised in Table 3. The first result quoted is based on the general solution in Table 2, it being found that three half-waves gives the minimum value of P (313 tons). With two half-waves, $P = 364$ tons, and with four half-waves, $P = 347$ tons. A cross-section of the truss in the deflected state when three half-waves form is shown in Fig. 4(b).

The analytical solutions given in Table 2 for the three cases ($A = 0, C_1 = C_2$), ($A = 0, C_1 = 0$) and ($C_1 = 0, C_2 = 0$) have been used in lines 2 to 4 of Table 3 to provide safe estimates of P. The values of the various flexural and torsional stiffnesses assumed in each case are quoted, as also are the buckling lengths l. It is interesting to note that the actual buckling length (20 feet, line 1 of Table 3) is smaller than any of those obtained in the subsequent safe estimates of P, being slightly less than the shortest (26 feet, line 4). The shortest buckling length obtained from the safe estimates may, in fact, always be used



WARREN TRUSS

(a) DIMENSIONS.

(b) DEFORMATION AT BUCKLING LOAD.

FIG. 4.

TABLE 3.

	ANALYSIS USED.	VALUES OF FLEXURAL AND TORSIONAL RIGIDITIES TONS - IN.²						BUCKLING LENGTH l FEET.	CRITICAL P TONS.
		A $10^3\times$	B_2 $10^3\times$	B_4 $10^3\times$	C_1 $10^3\times$	C_2 $10^3\times$	C_4 $10^3\times$		
1	GENERAL	150	645	73	220	496	56	20	313
2	$A = 0, \ C_1 = C_2$	0	645	73	220	220	56	60	172
3	$A = 0, \ C_1 = 0$	·0	645	73	0	496	56	41	173
4	$C_1 = 0, \ C_2 = 0$	150	645	73	0	0	56	26	176
5	ENGESSER	150	645	73	0	0	0	26	133

to obtain an approximation to the actual buckling length. Finally, if the Engesser formula is applied (line 5), allowance being made for the incomplete torsional restraint on the lower chord, then $P = 133$ tons. The Engesser formula thus underestimates the critical load in this case by some 57 per cent.

Application to Trusses with Non-Uniform Thrust in Compression Chord

The analysis contained in this article has been derived by reference to the case of uniform thrust in the compression chord. The " safe " results contained in Table 2 may, however, be used to obtain an approximate criterion of safety for trusses with non-uniform thrusts, and also non-uniform cross-sections, since these " safe " results do not involve the length between points of support. A truss will be stable provided the thrust P does not exceed the value given by the appropriate " safe " equations in Table 2 at any section along the truss. The member properties assumed in any such calculation are those corresponding to the particular section of the truss considered, certain properties being reduced until one of the cases contained in Table 2 becomes relevant.

An exception to the validity of the above criterion may occur when the axial thrusts in the web members become appreciable in relation to their individual buckling loads as pin-ended members. Hrennikoff[7] in his treatment considered the effect of such thrusts, but this is a subject requiring more detailed study. It may be noted that in many trusses, the effect of axial thrusts in some of the web members will be largely compensated by equal tensions in adjacent members. Moreover, the most critical section of the truss for buckling of the compression chord will usually correspond to that section where the axial loads in the web members are small. It may therefore be concluded that the effect of axial thrusts in web members may usually be neglected.

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Discussion

The Council would be glad to consider the publication of correspondence in connection with the above paper. Communications on this subject intended for publication should be forwarded to reach the Institution by the 30th July, 1960.

Book Reviews

Advanced Structural Design, by Cyril S. Benson. (London : Batsford, 1959). 9in. × 6in., 329 plus xiii pp. 50s.

This interesting and practical book consists essentially of the material one would compile in designing seventeen structures. Of these ten deal with structural steel projects, five with reinforced concrete structures and two with brickwork construction. The problems analysed are diverse : included are the complete designs for a grain silo, highway bridge, tank structure, 120 feet span shed, two-bay portal plant house, theatre balcony, multi-storey office building, bunkers, gantries and a chimney.

The work is clearly presented in a digestible form, and the calculations are skilfully augmented with explanation where necessary. It should help to broaden the structural horizons of the many designers who, unfortunately, are only trained in dealing adequately with one structural material and rarely get the chance to see what the other fellow does in practice.

Unfortunately, the book must lose a great deal of impact in that BS.449 : 1948 is used for the steelwork designs. It may also be said that the designs tend to be dated and many will regret that there is no reference to the Plastic Theory in steelwork, or to prestressed concrete and that little guidance is given to the student designing welded structures.

Despite these shortcomings, the book may prove a useful stimulant for students planning to take the Associate membership examination. R.H.

Civil Engineering Contracts Organization, by John C. Maxwell-Cook. (London : Cleaver-Hume Press, 1959). $8\frac{1}{2}$ in. × $5\frac{1}{2}$ in., 220 plus viii pp. 22s. 6d.

The Author sets out comprehensively the procedure from the inception of a scheme by the employer to its development in the contract stage, giving the relationships between the various parties concerned, advisory and executive, for the successful completion of a civil engineering project. A survey is given of the contract documents in general use with definitions of contract terms and useful comments on the clauses, and some piquant observations on financial arrangements.

The personnel required for the execution of the complete scheme is described from top level to the labourer and detailed as regards their duties and relationships. The importance to the client and contractor of a preliminary site survey and report by the consulting engineer on ground conditions has not, perhaps, been given sufficient place.

The chapters on specifications are perhaps not sufficiently up to date particularly regarding concrete which is mostly specified by quality control and strength nowadays, nor is there any mention of materials testing. Most large contracts are usually now equipped with a site laboratory for this purpose.

The site organization section contains many useful suggestions and aids to economy and smooth running of work on a construction site, and stresses the desire for design to be allied to easy and rapid execution at site. The Glossary forms a welcome appendix.

The book will be interesting and informative to all concerned in the development and execution of civil and structural engineering projects. F.T.B.

Linear Structural Analysis, by P. B. Morice, D.Sc., Ph.D., A.M.I.C.E., A.M.I.Struct.E. (London : Thames & Hudson, 1959). $9\frac{1}{2}$ in. × $6\frac{1}{4}$ in, 170 plus xii pp., 35s.

Since the 1930's, methods such as strain-energy and least work have steadily been giving way to methods of successive approximation for the analysis of many forms of elastic structures. Quite recently, however, the introduction of matrix algebra to structural analysis has greatly increased the usefulness of the classical approach. Moreover, electronic computers are available to perform the tedious numerical work involved in processes such as matrix inversion, thus making possible the solution of problems having a high degree of indeterminacy.

This book forms an excellent introduction to the subject. The first two chapters are largely devoted to the basic concepts of strain energy, Castigliano's theorems, and influence coefficients, and include a variety of examples. There follows a treatment of the question of the degree of indeterminacy of structures by a method which is stated to be without exception in its application to skeletal structures and, as such, represents a notable advance on previous methods.

Matrix algebra is then introduced, particular attention being paid to computational procedures, the description of each type of matrix operation being accompanied by a simple worked example. The following chapters deal with scale factors, transformation of co-ordinate systems and a number of points concerning release systems (of which the suitable choice may simplify the analysis). The final chapter deals very briefly with the programming of an electronic digital computer for structural analysis, and an appendix gives four worked examples which nicely illustrate the methods developed in the main text. There are numerous references which will help the student who wishes to pursue further any aspect of the subject.

E. M.